KFKI-71-76

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SYMMETRIES OF WIGNER COEFFICIENTS AND THOMAE - WHIPPLE FUNCTIONS

Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

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Submitted to Acta Physica
Hungarica

ABSTRACT

Wigner coefficients of the three-dimensional rotation group can be brought into the form of Thomae-Wipple functions. The symmetry group of order 72, discovered by Regge, is a straightforward consequence of 6 forms of the 120 Thomae-Whipple functions. The question whether the remaining 114 forms of these functions lead to new symmetries is investigated. It is shown that if the Regge group is enlarged by the transformations $j \to -j-1$, a group or order 1440 is obtained, which is exactly the group generated by the interrelations between the 120 Thomae-Whipple functions.

РЕЗЮМЕ

Коэффициенты Вигнера можно привести к виду функций Томе-Виппла. Наличие группы симметрий порядка 72, обнаруженной Редже, является непосредственным следствием шести разных видов I20 функций Томе-Виппла. Рассмотрен вопрос о том, приводят ли остальные II4 видов этих функций к новым свойствам симметрий коэффициентов Редже. Показано, что если группа Редже расширена с преобразованиями $j \to -j-1$, то получается группа порядка I440, которая производится с помощью соотношений между I20 функциями Томе-Виппла.

KIVONAT

A háromdimenziós forgáscsoport Wigner-együtthatói a Thomae-Whipple függvények alakjára hozhatók. A Regge által felismert 72 elemü szimmetriacsoport a 120 Thomae-Whipple függvény hat különböző alakjának következménye. A dolgozatban azt a kérdést vizsgáljuk, hogy az ezen függvények fennmaradó 114 alakja eredményez-e ujabb szimmetriát. Megmutatjuk, hogy amennyiben a Regge-csoportot $j \to -j-1$ alaku transzformációkkal bővitjük ki, egy 1440 elemü csoportot kapunk, amelyet éppen a 120 Thomae-Whipple függvény közt fennálló összefüggések generálnak.

1/ Following the derivation of the Wigner coefficients for the rotation group it was recognized early that these coefficients posses a symmetry group of twelve elements, six elements being the permutations of three angular momenta and the remaining ones combinations of a space reflection and the above permutations. It was discovered by Regge [1] that the Wigner coefficients are invariant under a larger symmetry group, namely, under a group of 72 elements. These symmetries are exhibited by the following table:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{bmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \end{bmatrix} \qquad /1/$$

The full symmetry group of the Wigner coefficients consists of the permutations of three columns and three rows of a reflection through the main diagonal. /To be strict, the table has to be multiplied by $(-1)^{P(j_1+j_2+j_3)}$, where P is the parity of the permuation./ Thus we get a group of $3!3!2! =$ $= 72$ elements, which will be denoted by R_{72}.

An explicit form for the Wigner coefficients can be obtained by integration of the product of three D-functions, by solving recurrence formulas satisfied by the Wigner coefficients, by expanding certain spinor invariants in power series [1], etc. The final result obtained by any of the above methods can be brought into a form of a generalized hypergeometric function ${}_3F_2$ of unit argument. The symmetry properties of these functions were derived a long time ago by Thomae and Whipple and, in fact, all 72 symmetries are straightforward consequences of these mathematical theorems.

2/ At first, by making use of a theorem by Burchnall and Chaundy, a simple analytic derivation for the Clebsh-Gordan coefficients will be presented. This derivation avoids rather cumbersome algebraic manipulations and gives the Clebsch-Gordan coefficients directly in a ${}_3F_2$ form, which is particularly apt for the investigation of symmetry properties. To this end consider representations of the rotation group in the form

$$D^j_{mn}(\varphi,\vartheta,\psi) = \eta e^{-i(m\varphi+n\psi)}\, n^j_{mn} \left(\sin\frac{\vartheta}{2}\right)^{m-n} \cdot \left(\cos\frac{\vartheta}{2}\right)^{m+n} \cdot$$

$$\cdot\, {}_2F_1\left(-j+m,\ j+m+1;\ m-n+1;\ \sin^2\frac{\vartheta}{2}\right) \qquad /2/$$

where

$$n^j_{mn} = \frac{1}{\Gamma(m-n+1)} \sqrt{\frac{\Gamma(j+m+1)\,\Gamma(j-n+1)}{\Gamma(j-m+1)\,\Gamma(j+n+1)}}$$

and η is a phase factor $\eta = i^{|m-n|+m-n}$, which for $m \geq n$ reduces to $\eta = (-1)^{m-n}$. Eq. /2/ is valid for $m \pm n \geq 0$. The remaining cases can be obtained by making use of symmetry properties of D-functions.

Clebsch-Gordan coefficients can be defined by the following decomposition:

$$D^{j_1}_{m_1 n_1}\, D^{j_2}_{m_2 n_2} = \sum_{j_3=|j_1-j_2|}^{j_1+j_2} C^{j_3 m_3}_{j_1 m_1; j_2 m_2}\, C^{j_3 n_3}_{j_1 n_1; j_2 n_2}\, D^{j_3}_{m_3 n_3} \qquad /3/$$

Recall a theorem by Burchnall and Chaundy [2] for the product of two ${}_2F_1$ hypergeometric functions:

$${}_2F_1(a,b;c;x)\ {}_2F_1(\alpha,\beta;\gamma;x) =$$

$$= \sum_{r=o}^{\infty} \frac{(a)_r\,(b)_r\,(\gamma)_r}{r!\,(c)_r\,(c+\gamma+r-1)_r}\ {}_3F_2\begin{bmatrix}\alpha,\ 1-c-r,\ -r\\ \gamma,\ 1-a-r\end{bmatrix}\ {}_3F_2\begin{bmatrix}\beta,\ 1-c-r,\ -r\\ \gamma,\ 1-b-r\end{bmatrix} \cdot \qquad /4/$$

$$\cdot\, x^r\ {}_2F_1(a+\alpha+r,\ b+\beta+r;\ c+\gamma+2r;x) \qquad \left((a)_r \equiv \frac{\Gamma(a+r)}{\Gamma(a)}\right)$$

Here ${}_3F_2$ is a generalized hypergeometric function of unit argument. In what follows it will be more convenient to work with functions introduced by Thomae and Whipple [3], instead of the $_3F_2$ functions. Using the identity x/ [3]

$$F_p(0;45) = \Gamma\begin{bmatrix}\alpha_{023}, & \alpha_{024}, & \alpha_{025}\\ \alpha_{123}, & \alpha_{124}, & \alpha_{125}\end{bmatrix} F_p(1;24) \qquad /5/$$

(α_{345} non-positive integer)

eq. /4/ can be rewritten as

$$_2F_1(a,b;c;x)\ _2F_1(\alpha,\beta;\gamma;x) = \sum_{r=o}^{\infty} \frac{(\alpha)_r\ (\beta)_r\ (c-a)_r\ (c-b)_r}{r!(c)_r\ (\gamma)_r\ (c+\gamma+r-1)_r}$$

$$\cdot\ _3F_2\begin{bmatrix}a, & \gamma-\alpha, & -r\\ 1+a-c-r, & 1-\alpha-r\end{bmatrix}\ _3F_2\begin{bmatrix}b, & \gamma-\beta, & -r\\ 1+b-c-r, & 1-\beta-r\end{bmatrix} x^r\ _2F_1(a+\alpha+r,\ b+\beta+r;\ c+\gamma+2r;x).$$

Before applying the theorem /6/ rewrite eq. /2/ in the form

$$D^j_{mn}(\varphi,\vartheta,\psi) = \eta(-1)^{j-m}\, e^{-i(m\varphi+n\psi)}\, n^j_{mn}\, \Gamma\begin{bmatrix}m-n+1, & 2j+1\\ j+m+1, & j-n+1\end{bmatrix} \qquad /7/$$

$$\left(\sin\frac{\vartheta}{2}\right)^{2j-m-n}\left(\cos\frac{\vartheta}{2}\right)^{m+n}\ _2F_1\left(-j+m,\ -j+n;\ -2j;\ \frac{1}{\sin^2\frac{\vartheta}{2}}\right)$$

The series form of the $_2F_1$ function entering eq. /2/ terminates. Here, in eq. /7/, the terms of the series are merely written in the reverse order.

On applying the theorem of Burchnall and Chaundy in form /6/ to the product of two D-functions given by eq. /7/ and futhermore changing

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x/

The abbreviation $\Gamma\begin{bmatrix}a,b,\ldots x\\ u,v,\ldots z\end{bmatrix} = \frac{\Gamma(a)\ \Gamma(b)\ldots\Gamma(x)}{\Gamma(u)\ \Gamma(v)\ldots\Gamma(z)}$ is used.

Further notations are those of Ref. [3].

the summation index to $j_3 = j_1+j_2-r$ one gets the required decomposition /3/. The Clebsch-Gordan coefficients obtained in this way are

$$C^{j_3 m_3}_{j_1 m_1; j_2 m_2} = \sqrt{2j_3+1}\;\Gamma\begin{bmatrix} j_1+m_1+1,\ j_2-m_2+1,\ j_3-m_3+1,\ j_3+m_3+1,\ j_1-j_2+j_3+1,\ -j_1+j_2+j_3+1 \\ j_1-m_1+1,\ j_2+m_2+1,\ j_1+j_2-j_3+1,\ j_1+j_2+j_3+2 \end{bmatrix}^{1/2}$$

$$\frac{1}{\Gamma(j_3-j_2+m_1+1,\ j_3-j_1-m_2+1)}\ {}_3F_2\begin{bmatrix} -j_1+m_1,\ -j_2-m_2,\ -j_1-j_2+j_3 \\ j_3-j_1-m_2+1,\quad j_3-j_2+m_1+1 \end{bmatrix} \qquad /8/$$

It is to be noted that identification of Clebsch-Gordan coefficients through eq. /3/ is not quite free of ambiguity, since C can contain an arbitrary real factor $\alpha = \alpha(j_1,j_2,j_3)$ of unit modulus. In eq. /8/ the value of α has been chosen in such a way that on the right hand side no power of (-1) should appear.

Wigner coefficients /1/ are related to the Clebsch-Gordan coefficients by

$$C^{j_3,-m_3}_{j_1 m_1; j_2 m_2} = (-1)^{j_1-j_2-m_3}\sqrt{2j_3+1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \qquad /9/$$

3/ Out of the 72 symmetries of the Wigner coefficients, 12 are simple consequences of formula /8/, as ${}_3F_2$ functions are invariant under permutations of the three numerator and two denominator parameters, which gives immediately $3!2! = 12$ symmetries. Thus, in order to obtain 72 symmetries, six different forms of ${}_3F_2$ function have to be found.

It is easy to see which types of transformations are required to get the six ${}_3F_2$ functions. In Regge table /1/ the sum of elements in any row or column is equal to $S = j_1+j_2+j_3$ and this sum remains unaltered by interchange of rows or columns, or by transposition through the main diagonal. Another characteristic of the ${}_3F_2$ function sum is closely related to S , as it can be seen from eq. /8/ that the sum of the denominator minus sum of the numerator parameters is $s = j_1+j_2+j_3+2 = S+2$. /This sum is responsible for the convergence of ${}_3F_2$ series when they contain an infinite number of terms. In the present case, however, no problem of convergence arises, since the ${}_3F_2$ series terminates./ Thus we have to search for such ${}_3F_2$ transformations which preserve the value of s. There are exactly six ${}_3F_2$ functions which have the above characteristic sum equal to s. These are, $F_p(0;45)$ /by definition this is proportional to the ${}_3F_2$ function entering eq. /8// $F_p(4;05)$, $F_p(5;04)$, $F_n(1;23)$, $F_n(2;13)$, $F_n(3;21)$. It is important that these functions, and indeed all the Thomae-Whipple functions, are attainable from $F_p(0;45)$. It can be verified that in such a way all the Regge symmetries are obtained.

4/ There exist 120 Thomae-Whipple functions and already six of them involve the R_{72} group. The question arises whether the remaining relations for these functions result in new symmetries for the Wigner coefficients. To see this, let us allow negative values of angular momenta. It can be seen from eq. /2/ that with the substitutions $j \to -j-1$ representations of the rotation group are merely multiplied by the phase $(-1)^{m-n}$. Hence, some of the angular momenta j_1, j_2, j_3 may be transformed into $j \to -j-1$. These transformations constitute a group of eight elements which fails to commute with Regge transformations, and therefore the enlarged Regge group G is not of order 8.72. To count the elements of G consider the subgroup which leaves the sum $S = j_1+j_2+j_3$ invariant. This subgroup is clearly the R_{72} group. The number of cosets in G with respect to the subgroup R_{72} is equal to the possible values of S. It can be seen by inspection that S can take the following values:

S	
$j_1+j_2+j_3$	1 of this type
$-j_1+j_2+j_3-1$	3x/ "
$-j_1-j_2+j_3-2$	3 "
$-j_1-j_2-j_3-3$	1 "
j_1-m_1-1	3 "
j_1+m_1-1	3 "
$-j_1-m_1-2$	3 "
$-j_1+m_1-2$	3 "

x/ The values of S not indicated here explicitly, can be obtained by permutation of the indices.

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Altogether 20 cosets are obtained, and thus if the R_{72} group is enlarged by transformations $j \to -j-1$ a group of order $20.72 = 1440$ is obtained.

Finally, we answer the question of how many different Thomae-Whipple functions are needed to cover this group of transformations when two such functions are considered to be different if they cannot be transformed into each other by permutations of numerator or denominator parameters. It is evident that $1440/12 = 120$ such functions are necessary, and these are the 120 Thomae-Whipple functions.

ACKNOWLEDGEMENTS

I would like to thank Prof. L.Jánossy for his interest to this work. I am grateful to Prof. Smorodinsky for discussions at an early stage of this work.

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Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Jánossy Lajos, a KFKI
Elméleti Kutató Csoportjának vezetője
Szakmai lektor: Király Péter
Nyelvi lektor: T. Wilkinson
Példányszám: 145 Törzsszám: 71-6178
Készült a KFKI sokszorosító üzemében,
Budapest, 1971. december hó